Is the QCD ghost dressing function finite at zero momentum ?

Philippe Boucaud,[a] **Thorsten Brüntjen,**[a] **Jean Pierre Leroy,**[a] **Alain Le Yaouanc,**[a] **Alexey Lokhov,**[b] **Jacques Micheli,**[a] **Olivier Pène**[a] **and Jose Rodriguez-Quintero**[c]

[a]*Laboratoire de Physique Théorique et Hautes Energies**
Université de Paris XI
Bâtiment 211, 91405 Orsay Cedex, France
[b]*Centre de Physique Théorique*† *de l'Ecole Polytechnique*
F91128 Palaiseau cedex, France
[c]*Dpto. Física Aplicada, Fac. Ciencias Experimentales*
Universidad de Huelva, 21071 Huelva, Spain
E-mail: `Philippe.boucaud@th.u-psud.fr`, `Thorsten.bruentjen@th.u-psud.fr`,
`Jean-Pierre.Leroy@th.u-psud.fr`, `Alain.Le-Yaouanc@th.u-psud.fr`,
`lokhov@cpht.polytechnique.fr`, `Jacques.Micheli@th.u-psud.fr`,
`Olivier.Pene@th.u-psud.fr`, `jose.rodriguez@dfaie.uhu.es`

ABSTRACT: We show that a finite non-vanishing ghost dressing function at zero momentum satisfies the scaling properties of the ghost propagator Schwinger-Dyson equation. This kind of Schwinger-Dyson solutions may well agree with lattice data and provides an interesting alternative to the widely spread claim that the gluon dressing function behaves like the inverse squared ghost dressing function, a claim which is at odds with lattice data. We demonstrate that, if the ghost dressing function is less singular than any power of p, it must be finite non-vanishing at zero momentum: any logarithmic behaviour is for instance excluded. We add some remarks about coupled Schwinger-Dyson analyses.

KEYWORDS: Lattice Gauge Field Theories, QCD, Lattice QCD.

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*Unité Mixte de Recherche 8627 du Centre National de la Recherche Scientifique
†Unité Mixte de Recherche 7644 du Centre National de la Recherche Scientifique

Contents

1. Introduction

The infrared behaviour of Landau gauge lattice gluon and ghost propagators is an interesting and hot subject. Two main methods are used: lattice QCD (LQCD) and Schwinger-Dyson equations (SDE) in which we include related methods as RGE, etc. In ref. [1] we have shown that a combination of both methods is extremely enlightening as it combines the advantages of lattice QCD's full control of errors and SDE's analytical character.

We only consider the particularly simple ghost propagator SDE :

We have studied the discrepancy between LQCD data and a widely spread belief: the ghost propagator SDE is claimed to imply a gluon dressing function behaving like the inverse squared ghost one. In ref. [1] we have reconsidered the scaling properties of the SDE and found three possible ways out of this problem (which are summarised in table 1 of that paper). The first one is to assume a singular behaviour of the ghost-gluon vertex in the deep infrared. A second possibility implies a very singular ghost dressing function which is excluded by LQCD. The third one is to assume that the ghost dressing function is less singular in the infrared that any power of p. In view of the general belief that the ghost dressing function was strongly singular we had not paid in ref. [1] attention to the third one.

Very recently, Sternbeck et al. [2] have produced two new evidences: i) the ghost-gluon vertex seems not to be singular, ii) the ghost dressing function seems to behave at most like $\log p$ in the infrared. These two evidences, taken together, strongly encourage us to consider now seriously the third above-mentioned solution. This is the aim of the present letter.

To our surprise we found that one can *demonstrate* from the scaling analysis of the ghost propagator SDE the impossibility of a $\log p$ behaviour or any other behaviour which is less divergent than any power of p: under these conditions, the ghost dressing function necessarily has a *finite non-vanishing* limit at zero momentum. This is at odds with a very general belief that the ghost dressing function is divergent. The proof will be displayed in section 3. We will shortly discuss published results about coupled gluon and ghost SDE in section 4.

2. Notations and summary of up-to-date lattice results

We use the following notations [1]:

$$
\begin{aligned}
\widetilde{\Gamma}_\mu(-q,k;q-k) &= q_\mu H_1(q,k) + (q-k)_\mu H_2(q,k) \\
\left(F^{(2)}\right)^{ab}(k^2) &= -\delta^{ab}\frac{F(k^2)}{k^2} \\
\left(G^{(2)}_{\mu\nu}\right)^{ab}(k^2) &= \delta^{ab}\frac{G(k^2)}{k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right),
\end{aligned}
\tag{2.1}
$$

where $G^{(2)}$ and $F^{(2)}$ are respectively the gluon and ghost propagators, G and F are respectively the gluon and ghost dressing functions and where $\widetilde{\Gamma}_\mu(-q,k;q-k)$ is the ghost-gluon vertex (k and q are the momenta of the incoming and outgoing ghosts and $q-k$ the momentum of the gluon) .

Following for simplicity the common, convenient, but not really justified, assumption of a power-law behaviour of the propagators in the deep infrared, we define

$$
F(k^2) \sim \left(\frac{k^2}{\nu}\right)^{\alpha_F}, \; G(k^2) \sim \left(\frac{k^2}{\lambda}\right)^{\alpha_G}.
\tag{2.2}
$$

In ref. [1] we have also defined an infrared exponent α_Γ for the vertex funtion H_1 ($\alpha_\Gamma < 0$ means a singular infrared behaviour).

Using the ghost propagator SDE equation it is often claimed that $2\alpha_F + \alpha_G = 0$. This belief is so strong that one often uses only one parameter $\kappa = -\alpha_F = \alpha_G/2$. However, as we will see in more details, everybody agrees that α_G is close [1] to 1 and it becomes now clear [2] that α_F is close to zero. Then *the relation $2\alpha_F + \alpha_G = 0$ is not satisfied [1, 2] and the arguments which support it have to be reconsidered.*

The lattice gluon propagator. Several SDE studies ([3] and references therein) predict a vanishing zero momentum propagator while, as discussed in ref. [4], a gluon propagator converging continuously to a non-zero value at vanishing momentum is a rather general

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[1]For example, in many SDE approaches it is found [3] $\alpha_G \simeq 1.18$

[2]One may wonder why many power law fits have given negative α_F. Our own fit in ref.[1] (table 2) has produced negative values very close to zero, but the errors were clearly underestimated. Presumably the systematic one, due to the functional form chosen for the fit, has not been properly taken into account. This is also the case in several other published results.
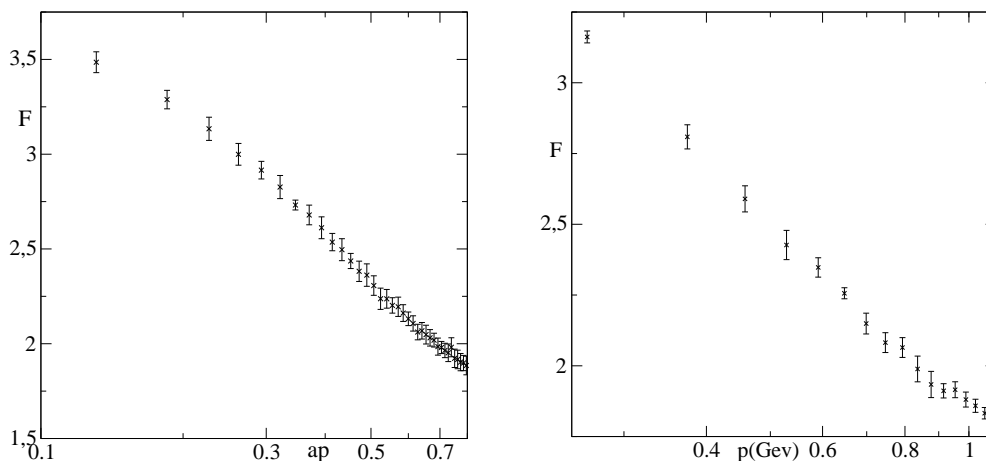
Figure 1: $F(p)$ from lattice simulations for SU(2) (left, $Vol = 48^4$, $\beta = 2.3$) and SU(3) (right, $Vol = 32^4$, $\beta = 5.8$). Logarithmic scales are used for x-axis in both plots. $\beta = 2.3$ for SU(2) has been chosen to guarantee that the string tension in lattice units is close to that of $\beta = 5.8$ for SU(3).

lattice result (in particular, the authors of ref. [5] obtain a non-vanishing infrared limit for the gluon propagator at a lattice volume of around 2000 fm^4). Therefore our preferred solution [3] is

$$\alpha_G = 1. \tag{2.3}$$

But, even if we relax this relation and assume a vanishing gluon propagator with $\alpha_G > 1$, the solution with a finite ghost dressing function at zero momentum remains possible as we shall see.

The lattice ghost propagator. The infrared behaviour of the ghost dressing function estimated from lattice simulations is a current controversial subject (see [7] and references therein). Very recent lattice estimates [2] seem to point towards a ghost dressing function rather close to the perturbative behaviour: the dressing function only shows, if any, a logarithmic dependence on the momentum (see ref. [2, figure 2]).

We confirm these results. In figure 1 the ghost dressing function is plotted as a function of $\log(p)$ for small values of the momenta. These plots were obtained from lattice simulations at $\beta = 5.8$ and a volume 32^4 in the SU(3) case and at $\beta = 2.3$ and a volume 48^4 in the SU(2) case. It is clear from these plots that $F(p)$ does not exhibit any power law: $\alpha_F = 0$. For SU(3) $F(p)$ is approximately linear in $\log(p)$ and for SU(2) it has even a smoother behavior (In this case one obtains a good fit of the data with a function $C(\log|p|)^\gamma$ and $\gamma \approx 0.4$).

The ghost-gluon vertex. In ref. [2], the authors did not find any evidence for a singularity in the case of a vanishing gluon momentum. Let us remark that this particular

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[3]Let us recall however that there is still a problem coming from the Slavnov-Taylor identity for the three gluon-vertex: we have shown that, the vertices being regular when one momentum tends to zero, it implies $\alpha_G < 1$. Is it possible to avoid any contradiction by assuming, as done by Cornwall [6], a non-regular behaviour for the longitudinal part of the three-gluon vertex? This deserves more investigation.

kinematical configuration isolates the form factor H_1 (see Eq.(2.1)) which enters in the SD equation (It is worth recalling that in perturbation theory $H_1(q, 0) + H_2(q, 0)$ is equal to 1 although $H_1(q, 0)$ is not [8]). Our dimensional analysis of the ghost SDE (see next section 3) invokes a different kinematical configuration for this form factor, $H_1(q, k \to 0)$ instead of $H_1(k, k)$. The non-singular behaviour they found as k tends to 0 excludes however the singularity of the ghost-gluon vertex that we proposed in ref. [1] as our favoured solution to reconcile the ghost propagator SDE and the lattice inspired relation $2\alpha_F + \alpha_G > 0$.

In conclusion lattice simulations show a strong evidence that $\alpha_G = 1$ and $2\alpha_F + \alpha_G > 0$, far from zero. Now we have a fair indication that $\alpha_F = 0$ and about the regularity of the vertex form factor involved in the ghost SDE. This leads us to revisit, in next section, the case in column 4 of table 1 in ref. [1] for ghost and gluon propagators and vertices satisfying the scaling properties of ghost SDE.

3. Ghost SDE: the case $\alpha_F = 0$

We will now demonstrate that $F(0)$ is finite non-vanishing for $\alpha_F = 0$. We will exploit the constraints, summarized in table 1 of ref. [1], between the infrared exponents α_F, α_G, α_Γ, from the ghost SDE. The IR convergence of the loop integral in the ghost SDE implies the two conditions:

$$\alpha_F + \alpha_\Gamma > -2 \ , \ \alpha_G + \alpha_\Gamma > -1 \ . \tag{3.1}$$

Then the dimensional consistency of ghost SDE at small momenta leads to only three allowed cases:

    i) $\alpha_F \neq 0$ and $\alpha_F + \alpha_G + \alpha_\Gamma < 1 \Longrightarrow 2\alpha_F + \alpha_G + \alpha_\Gamma = 0$

    ii) $\alpha_F \neq 0$ and $\alpha_F + \alpha_G + \alpha_\Gamma \geq 1 \Longrightarrow \alpha_F = -1$

    iii) $\alpha_F = 0$ and $\alpha_G + \alpha_\Gamma \geq 1$ does not require any further constraint.

We shall look, in the following, at the consequences of the third case. It includes in particular $\alpha_G = 1$ and $\alpha_\Gamma = 0$ which is favoured by lattice simulations (see section (2)). Nevertheless we shall not suppose, in the following derivation, anything more than $\alpha_G + \alpha_\Gamma \geq 1$ and conditions (3.1). This leaves open, for example, the possibility that the gluon propagator goes to zero in the IR limit, the vertex remaining finite or singular.

Of course, even with $\alpha_F = 0$, we cannot exclude *a priori* the possibility that $F(k)$ diverges or tends to zero more slowly than any power of k when $k \to 0$. We shall however prove that this is not allowed: $F(k)$ remains finite in this limit provided that the two following conditions are satisfied:

$$\alpha_F = 0 \ , \ \alpha_G + \alpha_\Gamma \geq 1 \tag{3.2}$$

Writing the subtracted bare SD equation for two scales λk and $\kappa \lambda k$ (see ref. [1, eq. (14)]) one obtains:

$$\frac{1}{F(\lambda k)} - \frac{1}{F(\kappa \lambda k)} = g_B^2 N_c \int \frac{d^4 q}{(2\pi)^4} \left(\frac{F(q^2)}{q^2} \left(\frac{(k \cdot q)^2}{k^2} - q^2 \right) \right)$$

$$\times \left[\frac{G((q-\lambda k)^2)H1(q,\lambda k)}{(q-\lambda k)^2} - \frac{G((q-\kappa\lambda k)^2)H1(q,\kappa\lambda k)}{(q-\kappa\lambda k)^2} \right] \right) , \qquad (3.3)$$

where λ is a parameter which we shall use to study the IR ($\lambda \to 0$) dimensional behaviour of F; κ is a fixed number, $0 < \kappa < 1$, needed to write a subtracted equation ensuring the UV convergence. It was shown in ref. [1] that the r.h.s. of eq. (3.3) is the sum of two terms behaving respectively as $\lambda^{2\mathrm{Min}(\alpha_F + \alpha_G + \alpha_\Gamma, 1)}$ and λ^2 when $\lambda \to 0$. So it behaves as λ^2 when the conditions (3.2) are satisfied. For any κ there is a value of λ and c such that $\forall \lambda' \leq \lambda$ we have $|\frac{1}{F(\lambda' k)} - \frac{1}{F(\kappa\lambda' k)}| \leq c\lambda'^2$, thus:

$$\left| \frac{1}{F(\lambda k)} - \frac{1}{F(\kappa\lambda k)} \right| \leq c\lambda^2$$

$$\vdots$$

$$\left| \frac{1}{F(\kappa^{n-1}\lambda k)} - \frac{1}{F(\kappa^n \lambda k)} \right| \leq c\lambda^2 \kappa^{2(n-1)} \qquad (3.4)$$

which implies:

$$\left| \frac{1}{F(\lambda k)} - \frac{1}{F(\kappa^n \lambda k)} \right| \leq c \frac{1 - \kappa^{2n}}{1 - \kappa^2} \lambda^2. \qquad (3.5)$$

So $F \to \infty$ when $\lambda \to 0$ is excluded because taking the limit of the above expression when $n \to \infty$ we should have $|\frac{1}{F(\lambda k)}| \leq c \frac{1}{1-\kappa^2} \lambda^2$ and F would diverge as or more rapidly than $\frac{1}{\lambda^2}$ implying $\alpha_F \leq -1$ in contradiction with the hypothesis $\alpha_F = 0$. Let us remark that $F \to 0$ is also excluded: Eq. (3.5) implies $|\frac{1}{F(\kappa^n \lambda k)}| \leq |\frac{1}{F(\lambda k)}| + c\frac{1-\kappa^{2n}}{1-\kappa^2}\lambda^2$ and $\frac{1}{F(\kappa^n \lambda k)}$ cannot tend to infinity when $n \to \infty$. This completes the proof. Notice that we have used bare Green functions and couplings, everything remains however exactly the same if we replace them by renormalized ones.

This is our main result: If $\alpha_F = 0$ the ghost dressing function has to be finite and $\neq 0$ in the IR limit. This solution is compatible with our knowledge from lattice simulations about the behavior of the ghost dressing function and ghost-gluon vertex. Of course, the current lattice simulations cannot yet exclude a smooth divergence which the preceding dimensional analysis forbids. A detailed numerical study of the ghost propagator in the deep IR is strongly needed.

4. Remarks about coupled gluon and ghost SDE solutions

The combination of the scaling analysis of ghost SDE and lattice predictions appears to be very restrictive concerning the low-momentum behaviour of gluon and ghost propagators. Such a behaviour must be a solution of the combined SDE for both gluon and ghost propagators. The schemes followed to solve the combined SDE's have often led to $2\alpha_F + \alpha_G = 0$ and $\alpha_G \gtrsim 1$ (hence a strongly divergent ghost dressing function). However, a two-loop analysis [9] proves to be much less restrictive in constraining α_G. Our findings require to reconsider these approaches by taking into due account the special case $\alpha_F = 0$.

In a recent paper [10] Aguilar and Natale found $\alpha_G = 1$ and $\alpha_F = -0.04$, not far from our present conclusions and deserving a closer comparison. They followed the Cornwall [6]

lattice	m_0 (GeV)	$Z_b(a)$
5.6 (24^4)	0.523(2)	3.69(1)
5.7 (32^4)	0.527(1)	3.85(1)
5.8 (24^4)	0.493(7)	3.88(3)
6.0 (32^4)	0.503(4)	3.99(2)
6.0 (24^4)	0.461(16)	3.97(6)

Table 1: Best-fit parameters.

prescription for the trilinear gluon vertex and solved the coupled equations for the ghost and gluon propagators in the Mandelstam approximation [11]. Concerning the ghost dressing function, in spite of the fact that they find it slowly power-like divergent, it remains flat till very small momenta. This last point is in contradiction with the lattice results in ref. [2] and ours in figure 1, where $F(k)$ is not at all so flat and shows a logarithmic enhancement as the momentum decreases. Of course, if power-like divergences are excluded, the arguments presented in section 3 imply a flat dressing function in a small momentum range presumably not yet reached by current lattice analyses.

To compare quantitatively their gluon propagator with LQCD [12], we have applied the simple parametrization they proposed:

$$G^{(2)}_{\text{bare}}(q^2; a, L) = \frac{Z_b(a)}{q^2 + \frac{m_0(L)^4}{q^2 + m_0(L)^2}} + \mathcal{O}(a, 1/L) \ , \tag{4.1}$$

where a stands for the lattice spacing and L for the lattice length. In refs. [12], the gluon propagator was estimated from 24^4 lattices at $\beta = 5.6, 5.8, 6.0$ and 32^4 lattices at $\beta = 5.7$ and $\beta = 6.0$ and analyzed through OPE or instanton liquid models that failed in describing the very low momentum range ($q < 0.4$GeV). In figure 2, we plot the curves corresponding to the best-fit parameters m_0 and Z_b collected in table 1. The parametrization eq. (4.1) matchs pretty well the lattice data [4]. Moreover one knows that, at the leading log,

$$d(\log(Z_b(a))) = \frac{13}{22} d(\log \beta) \ . \tag{4.2}$$

Performing a linear fit of $\log(Z_b)$ as a function of $\log(\beta)$ for $\beta \geq 5.7$ we obtain a slope approximately equal to 0.69 which has to be compared to $\frac{13}{22} = 0.59$. That result is unexpectedly good for the large lattice spacings we take in consideration.

5. Conclusions

The main result we presented in this brief note was to emphasize the interest of a general class of SDE solutions where $\alpha_F \approx 0$. This solution has the advantage of being compatible with other convincing lattice results namely: $\alpha_G = 1$ and $2\alpha_F + \alpha_G > 0$. It is also

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[4]The masses we obtain differ from the one quoted in ref. [10] but these depend on a parameter, Λ, which in their approach can be varied.
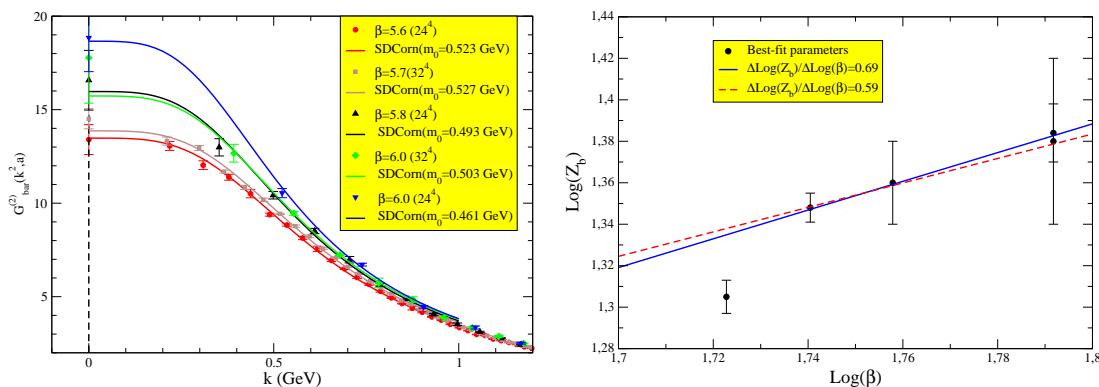
Figure 2: Best fits to the lattice data (left) and Log-log plot of Z_b in terms of the lattice bare gauge coupling parameter β (right). The solid blue (dotted red) line shows a fit to a linear formula where the slope is free to be fitted (fixed by one-loop perturbation theory in eq. (4.2)).

compatible with a still uncertain result concerning the regularity of the ghost-gluon vertex. We have proven that if $\alpha_F = 0$ the ghost dressing function must be finite in the IR limit. Of course one would need measures on larger volumes in order to test the finiteness of the ghost propagator in the limit $k \to 0$.

We have discussed some results from published coupled ghost and gluon SDE solutions and also shown that the lattice gluon propagator data at low momenta can be well described by the very simple parametrisation eq. (4.1) inspired by a recent gluon SDE analysis.

Acknowledgments

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